Tutorial class 6/4

1 Series of real numbers

Example 1.1. *Find the value of a such that the series*

$$\sum_{k=1}^{\infty}\left(\frac{1}{n}-\sin\frac{1}{n}\right)^{a} \quad \textit{exists.}$$

Proof. Consider $f(x)=\sin x$ on $[0,1]$. By Taylor's theorem, for $x>0$, there exists $\eta\in(0,x)$ such that

$$\sin x = f(x) = \sum_{k=0}^{3}\frac{f^{(k)}(0)}{k!}x^k + \frac{f^4(\eta)}{4!}x^4 = x - \frac{x^3}{3!} + \frac{x^4}{4!}\sin\eta.$$

Therefore,

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \frac{1}{6}.$$

Thus, there exists $\delta>0$ such that for all $0<x<\delta$,

$$\frac{1}{8}x^3 \le x-\sin x \le \frac{1}{2}x^3.$$

Hence by comparison test, the series exists if and only if $a>1/3$. □

2 Series of functions

Definition 2.1. *We say that $\sum_{k=1}^{\infty} f_k(x)$ converge if for each $x_0\in A$, the series of real number $\sum_{k=1}^{\infty} f_k(x_0)$ converge.*
It converges uniformly if its partial sum converge uniformly on A as a sequence of function.

(Cauchy Criterion) Equivalently, that is to say $\forall\epsilon>0$, there exists $N\in\mathbb{N}$ such that for all $m,n>N$, for all $x\in A$,

$$\Big|\sum_{k=n}^{m} f_k(x)\Big| < \epsilon.$$

One may need to discuss the continuity of limit functions.

Example 2.2. $f(x)=\displaystyle\sum_{n=1}^{\infty}\frac{\cos(3^n x)}{2^n}$ *is a well defined continuous function on $\mathbb{R}$.*

Proof. Since $\frac{\cos(3^n x)}{2^n}\le\frac{1}{2^n}$ for all $n\in\mathbb{N}$, by comparison test, $f(x)$ exists for all $x\in\mathbb{R}$. In order to show the continuity, we may need to show that the convergence is uniform. (Uniform convergence preserves the continuity.) The uniform convergence is immediate from the the control of 2^{-n}. Hence it is continuous. □

Remarks: Noted that uniform convergence is global, while continuity is local. Therefore, the uniform convergence assumption is probably overuse. See the example below.

Example 2.3. $f(x) = \sum_{n=1}^{\infty} \frac{e^{nx}}{n!}$ *is a continuous function, but the convergence is non-uniform.*

Proof. For each $x \in \mathbb{R}$, by ratio test

$$\frac{e^{(n+1)x}}{(n+1)!} \cdot \frac{n!}{e^{(n+1)x}} = \frac{e^x}{n+1} \to 0 \text{ as } n \to \infty.$$

Hence $f(x)$ is well defined. To show the continuity, it suffices to prove that f is continuous at $c \in \mathbb{R}$, where c is arbitrarily chosen from the real line.
Let $c \in \mathbb{R}$, $c \in [-M, M]$ where $M >> c$ or $-M << c$. Therefore it suffices to show that the convergence is uniform on $[-M, M]$.
Let $\epsilon > 0$, there exists N such that for all $m, n > N$,

$$|\sum_{k=n}^{m} \frac{e^{kx}}{k!}| \leq \sum_{k=n}^{m} \frac{e^{kM}}{k!} < \epsilon.$$

The first inequality hold whenever $x \in [-M, M]$ while the second one can be deduced from the convergence of such series. So f is continuous on $[-M, M]$, in particular at c.

To show the non-uniform issue, one observe that $\frac{e^{kx}}{k!}$ does not converge to 0 uniformly. It can be shown by picking $x_k = k$ for integer k. Then

$$\frac{e^{kx_k}}{k!} = \frac{e^{k^2}}{k!} \to +\infty \text{ as } k \to \infty.$$

In general, if $\sum f_k$ converges uniformly, then f_k converges to 0 uniformly since we have

$$f_n(x) = \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{n-1} f_k(x) \rightrightarrows 0.$$

□

Differentiability of the limit function is argued in a similar manner.

Example 2.4. $F(x) = \sum_{n=1}^{\infty} \frac{n^{10}}{x^n}$ *is differentiable on* $(1, \infty)$.

Proof. By the theory in sequence of functions, it suffices to show that $\sum_{n=1}^{\infty} \frac{n^{10}}{x^n}$ converges for some x_0 and also $\sum_{n=1}^{\infty} \frac{n^{11}}{x^{n+1}}$ converge uniformly around a fixed point c where c is arbitrarily chosen in $(1, \infty)$. The argument is completely the same as before. We leave it to the reader. □